On the Existence of Hamiltonian Paths in the Cover Graph of $M(n)$

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Abstract

The poset $M(n)$ has as its elements the n-tuples of integers $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ satisfying $0 = a_1 = \cdots = a_j < a_{j+1} < \cdots a_n \leq n$ for some j, $0 \leq j \leq n$. The order relation is defined by $\mathbf{a} \leq \mathbf{b}$ iff $a_i \leq b_i$ for $1 \leq i \leq n$. We show that the cover graph of $M(n)$ has a Hamiltonian path if and only if $\binom{n+1}{2}$ is odd and $n \neq 5$.

Keywords: Hamiltonian path, Gray code, cover graph, augmentation poset

1 Introduction

Introduced by Stanley [9], the poset $M(n)$ has as its elements the n-tuples of integers $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ satisfying $0 = a_1 = \cdots = a_j < a_{j+1} < \cdots a_n \leq n$ for some $j \geq 0$. The order relation is defined by $\mathbf{a} \leq \mathbf{b}$ iff $a_i \leq b_i$ for $1 \leq i \leq n$. A discussion of order properties of posets appears in [10]; a nice description of $M(n)$ in particular appears in [6]. Another description of $M(n)$ is as the lattice of order ideals in the product of chains of

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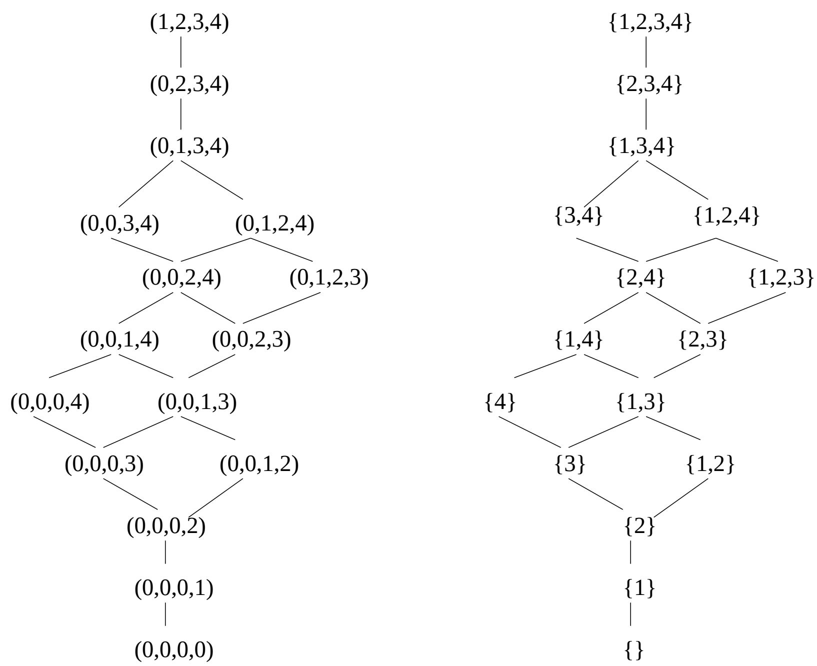

Figure 1: Two views of the cover graph of $M(4)$.

sizes 2 and $n-1$. The poset $M(n)$ is both ranked and order-symmetric. It has the Sperner property, and it is rank-unimodal [9, 6]. It is unknown whether $M(n)$ has a symmetric chain decomposition.

In contrast to the order-theoretic properties of $M(n)$, in this paper we consider graph-theoretic properties of its Hasse diagram. When viewed as an undirected graph, the Hasse diagram is called the *cover graph* of the poset. Vertices are adjacent in the cover graph if they are related elements of the poset with no other elements between them. As illustrated in Figure 1, the elements of $M(n)$ can be viewed as the subsets of the set $[n] = \{1, 2, \ldots, n\}$. In this phrasing, two vertices are adjacent in the cover graph if one is obtained from the other by adding the element 1 or by augmenting one element by one. We prove that the cover graph of $M(n)$ has a Hamiltonian path if and only if $\binom{n+1}{2}$ is odd and $n \neq 5$. We set up the main result in Section 2 and describe the construction in Section 3.

For comparison and motivation, we mention a more familiar partial order on the same set: the inclusion relation. This defines a poset on the subsets of $[n]$, called the *Boolean lattice* $B(n)$. Its cover graph is the n-cube. The n-cube is Hamiltonian and has Hamiltonian paths satisfying a wide variety of constraints (e.g. [2, 3, 4, 5, 7]). It is easy to obtain a Hamiltonian path in $B(n)$ by induction. The cover graph of $M(n)$ has a somewhat more complicated structure than that of $B(n)$ and fewer edges: $(n+1)2^{n-2}$ instead of $n2^{n-1}$.

2 Necessary Conditions

Let A_n be the cover graph of $M(n)$. For compactness, we will represent the vertices of A_n as subsets of $[n]$. We write element $\mathbf{a} \in M(n)$ satisfying $0 = a_1 = \cdots = a_j < a_{j+1} < \cdots a_n \leq n$ as the vertex $\{a_{j+1}, \ldots, a_n\}$ of A_n, as in Figure 1.

For sets X, Y, let $X \oplus Y = (X \cup Y) \setminus (X \cap Y)$. In A_n vertices X and Y are adjacent if and only if

(i) $|X| = |Y|$ and $X \oplus Y = \{i, i+1\}$ for some i or

(ii) $X \oplus Y = \{1\}$.

For a finite set S, let $\sigma(S)$ denote the sum of the elements of S. Note that if X and Y are adjacent in A_n then $\sigma(X)$ and $\sigma(Y)$ differ by 1. Thus A_n is bipartite with bipartition (E_n, O_n), where

$$E_n = \{X \in 2^{[n]} \ : \ \sigma(X) \text{ is even}\}$$
$$O_n = \{X \in 2^{[n]} \ : \ \sigma(X) \text{ is odd}\}$$

Proposition 1 *For $n \geq 1$, if $\binom{n+1}{2}$ is even, then A_n does not have a Hamiltonian path.*

Proof. Always $|E_n| = |O_n|$, so a Hamiltonian path in A_n must have one endpoint in E_n and the other in O_n. On the other hand, A_n has two vertices of degree 1, namely $\emptyset$ and $[n]$, which therefore must be the endpoints of any Hamiltonian path. If $\sigma([n]) = \binom{n+1}{2}$ is even, then the vertices that must be the endpoints both lie in E_n. 2

When $n = 5$, $\binom{n+1}{2}$ is odd, but A_5 does not have a Hamiltonian path. To see this, note in Figure 2 that vertices of degree 2 in A_n successively force a Hamiltonian path in A_5 to use the edges on the disjoint paths (shown in boldface on the figure):

$$\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{3\}, \{4\}, \{5\}, \{1,5\}, \{1,4\}, \{2,4\}, \{2,3\}, \{1,2,3\}, \{1,2,4\}$$

and

$$\{1,2,3,4,5\}, \{2,3,4,5\}, \{1,3,4,5\}, \{3,4,5\}, \{2,4,5\}, \{1,2,4,5\}, \{1,2,3,5\},$$
$$\{1,2,3,4\}, \{2,3,4\}, \{2,3,5\}, \{1,3,5\}, \{1,4,5\}, \{4,5\}, \{3,5\}.$$

However, the subgraph of A_n induced by $\{1,2,4\}$, $\{3,5\}$, $\{2,5\}$, $\{3,4\}$, $\{1,2,5\}$, and $\{1,3,4\}$ is a 6-cycle in which $\{1,2,4\}$ and $\{3,5\}$ are diametrically opposite. Hence the path cannot be completed.

In the remainder of this paper, we prove the following theorem.

Theorem 1 *For $n \geq 1$, if $\binom{n+1}{2}$ is odd and $n \neq 5$, then A_n has a Hamiltonian path.*

This is clear for $n \leq 2$. For $n > 5$, the result will follow from Lemma 5 in the next section.

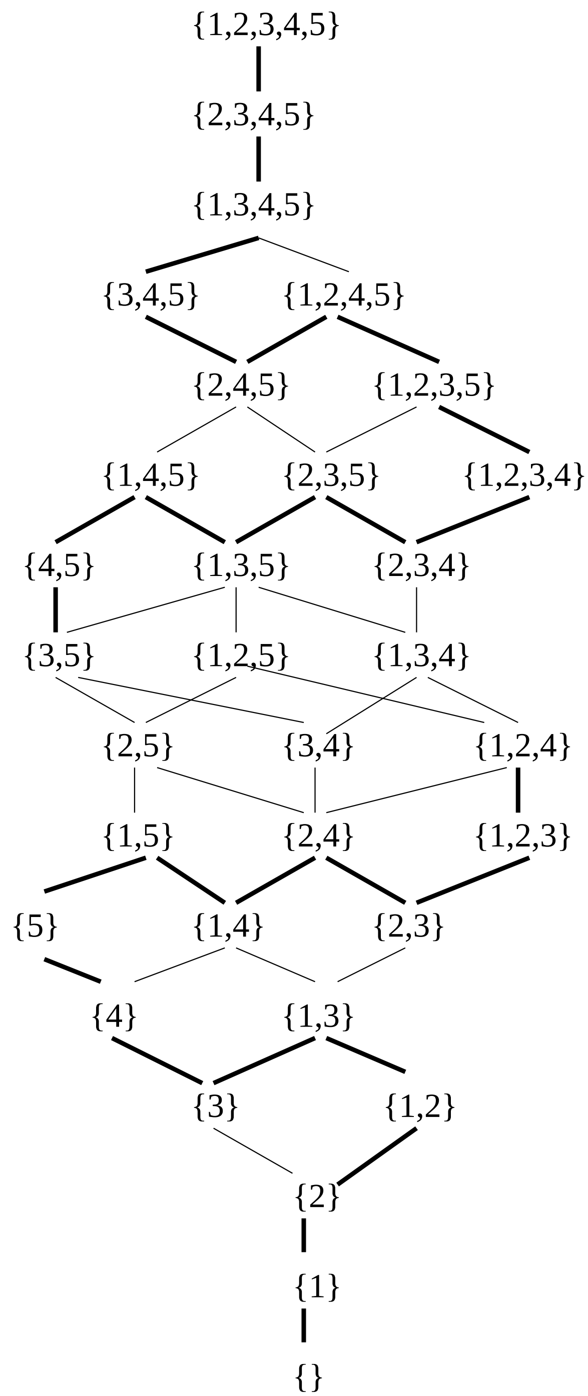

Figure 2: A_5, the cover graph of $M(5)$.

3 Construction of Paths

For $n \geq 3$, let G_n be the subgraph of A_n induced by the vertices $V(G_n) = V(A_n) \setminus \{\emptyset, \{1\}, \{2,3,\ldots,n\}, [n]\}$. Note that A_n has a Hamiltonian path if and only if G_n has a Hamiltonian path from $\{2\}$ to $\{1,3,\ldots,n\}$. We will prove via a sequence of lemmas that such a path exists in G_n whenever $\binom{n+1}{2}$ is odd and $n \neq 5$. Our strategy is to build up paths and cycles in subgraphs of G_n that we will combine to form the desired Hamiltonian path.

For $n \geq 3$, let e_n and a_n denote the edges in G_n:

$$e_n = \{1,2,\ldots,n-1\}\{2,\ldots,n-1\}$$

$$a_n = \{n\}\{1,n\}.$$

(We use uv to denote the edge joining adjacent vertices u and v.)

Lemma 1 *For $n \geq 3$, every Hamiltonian cycle in G_n contains the following edges:*

(a) $\{2\}\{3\}$

(b) $\{2\}\{1,2\}$

(c) $\{1,2\}\{1,3\}$

(d) $\{n\}\{1,n\} = a_n$

(e) $[n-1]\{2,3,\ldots n-1\} = e_n$.

Furthermore, for each edge uv in (a)-(e) every Hamiltonian path in G_n that does not start or end at u or v contains the edge uv.

Proof. This holds because each of the vertices $\{1,2,\ldots n-1\}, \{n\}, \{2\}$, and $\{1,2\}$ has degree 2 in G_n. *2*

When H is a subgraph of G_{n-1}, let nH denote the subgraph of G_n obtained from the graph H by replacing each vertex X of H by the vertex $X \cup \{n\}$. For an edge $e = XY$, let $V(e)$ denote the set $\{X, Y\}$. Then the vertex set of G_n is the disjoint union

$$V(G_n) = V(G_{n-1}) \cup V(nG_{n-1}) \cup V(e_n) \cup V(a_n). \tag{1}$$

Let P be a path in G_n containing edge uv, and let $e = wz$ be an edge of G_n with neither w nor z on P. If uw and vz are also edges of G_n, then by "*pulling e into P*" we will mean replacing the edge uv of P by the path u, w, z, v to obtain a new path P' with $V(P') = V(P) \cup V(e)$.

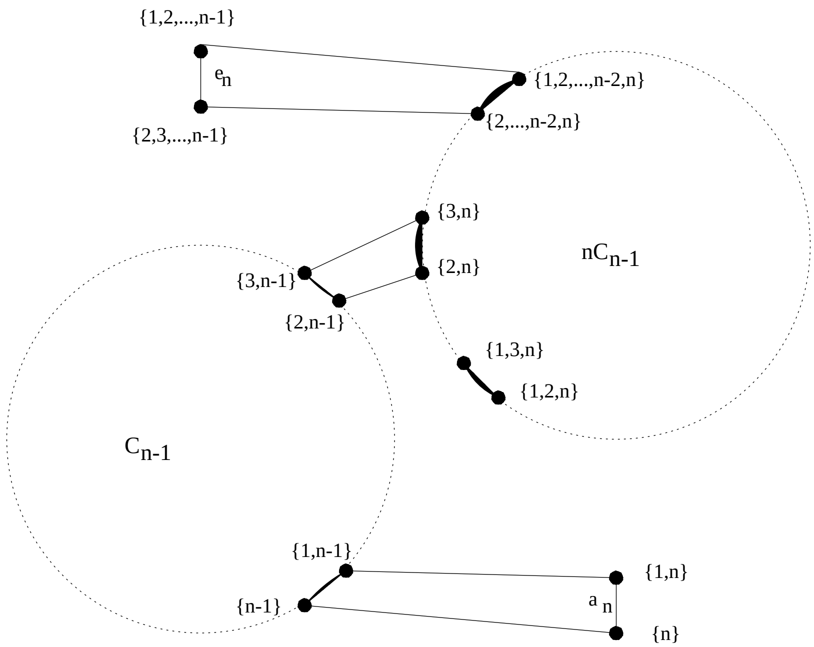

Figure 3: Construction of C_n in Lemma 2 when C_{n-1} contains $\{2, n-1\}\{3, n-1\}$.

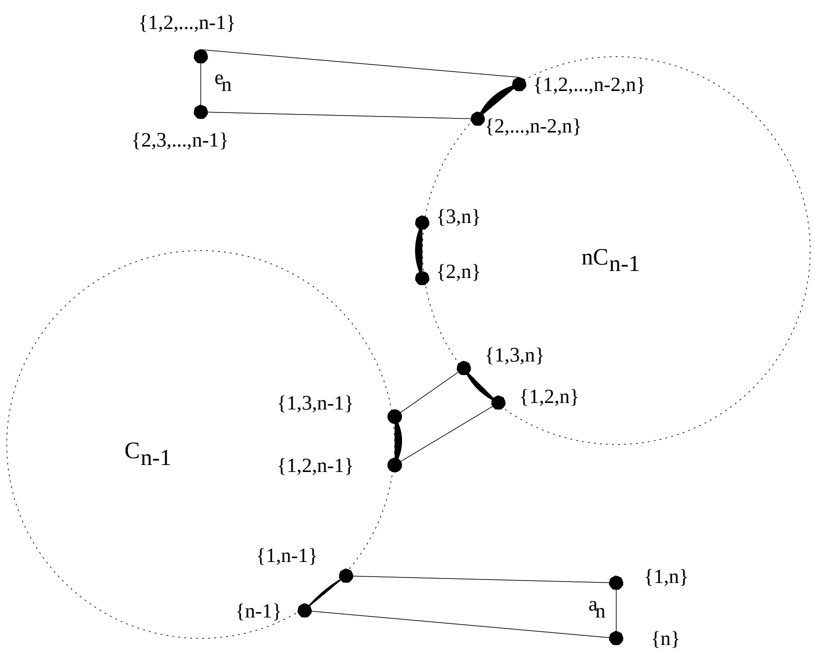

Figure 4: Construction of C_n in Lemma 2 when C_{n-1} contains $\{1, 2, n-1\}\{1, 3, n-1\}$.

Note that if P and Q are Hamiltonian paths in G_{n-1}, then by Lemma 1 we can pull a_n into P as long as P does not start or end with $\{n-1\}$ or $\{1, n-1\}$ and we can pull e_n into nQ as long as Q does not start or end with $[n-2]$ or $\{2, 3, \ldots, n-2\}$.

Lemma 2 *For $n \geq 3$, G_n has a Hamiltonian cycle C_n. For $n \geq 4$, there is such a cycle containing the edge $\{2, n\}\{3, n\}$ and such a cycle containing the edge $\{1, 2, n\}\{1, 3, n\}$.*

Proof. For $n = 3$, G_3 itself is the cycle C_3 (of length 4). For $n = 4$, the cycle C_4 below contains both special edges.

$$\{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{3, 4\}, \{2, 4\}, \{1, 4\}, \{4\}, \{3\}$$

Assume that $n > 4$, and let C_{n-1} be a Hamiltonian cycle in G_{n-1} satisfying one of the claimed conditions of the theorem. By (1), we can form C_n from C_{n-1} and nC_{n-1} as follows. Pull edge e_n into nC_{n-1} and edge a_n into C_{n-1} and then link the cycles together as described below.

If C_{n-1} contains edge $\{2, n-1\}\{3, n-1\}$, then delete this edge from C_{n-1} and delete edge $\{2, n\}\{3, n\}$ from nC_{n-1} (guaranteed to exist by Lemma 1). Add edges $\{2, n-1\}\{2, n\}$ and $\{3, n-1\}\{3, n\}$. (See Figure 3. In this and other figures vertices need not occur on cycles and paths in the order shown.) The resulting cycle contains edge $\{1, 2, n\}\{1, 3, n\}$, since Lemma 1 implies that C_{n-1} contains edge $\{1, 2\}\{1, 3\}$.

In the other case, C_{n-1} contains the edge $\{1, 2, n-1\}\{1, 3, n-1\}$. Delete this edge from C_{n-1}

$Q_5 =$

$\{2\}$,	$\{1,2\}$,	$\{1,3\}$,	$\{3\}$,	$\{4\}$,	$\{5\}$,	$\{1,5\}$,
$\{1,4\}$,	$\{2,4\}$,	$\{2,3\}$,	$\{1,2,3\}$,	$\{1,2,4\}$,	$\{1,3,4\}$,	$\{3,4\}$,
$\{3,5\}$,	$\{4,5\}$,	$\{1,4,5\}$,	$\{2,4,5\}$,	$\{3,4,5\}$,	$\{1,3,4,5\}$,	$\{1,2,4,5\}$,
$\{1,2,3,5\}$,	$\{1,2,3,4\}$,	$\{2,3,4\}$,	$\{2,3,5\}$,	$\{1,3,5\}$,	$\{1,2,5\}$,	$\{2,5\}$

Figure 5: The Hamiltonian path Q_5 in G_5 (read across) in Lemma 3.

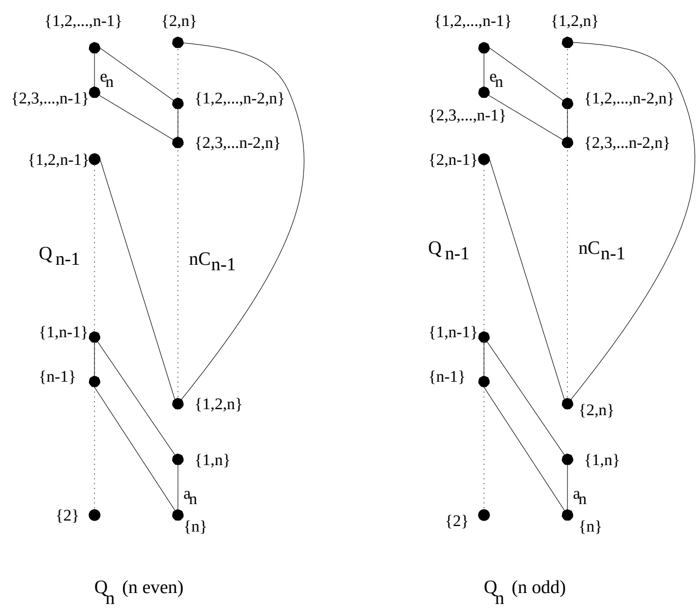

Figure 6: Construction of Q_n in Lemma 3.

$Q_5^* =$

$\{1,2\}$,	$\{2\}$,	$\{3\}$,	$\{4\}$,	$\{5\}$,	$\{1,5\}$,	$\{1,4\}$,
$\{1,3\}$,	$\{2,3\}$,	$\{1,2,3\}$,	$\{1,2,4\}$,	$\{1,3,4\}$,	$\{3,4\}$,	$\{2,4\}$,
$\{2,5\}$,	$\{3,5\}$,	$\{4,5\}$,	$\{1,4,5\}$,	$\{2,4,5\}$,	$\{3,4,5\}$,	$\{1,3,4,5\}$,
$\{1,2,4,5\}$,	$\{1,2,3,5\}$,	$\{1,2,3,4\}$,	$\{2,3,4\}$,	$\{2,3,5\}$,	$\{1,3,5\}$,	$\{1,2,5\}$

Figure 7: The Hamiltonian path Q_5^* in G_5 (read left to right) in Lemma 4.

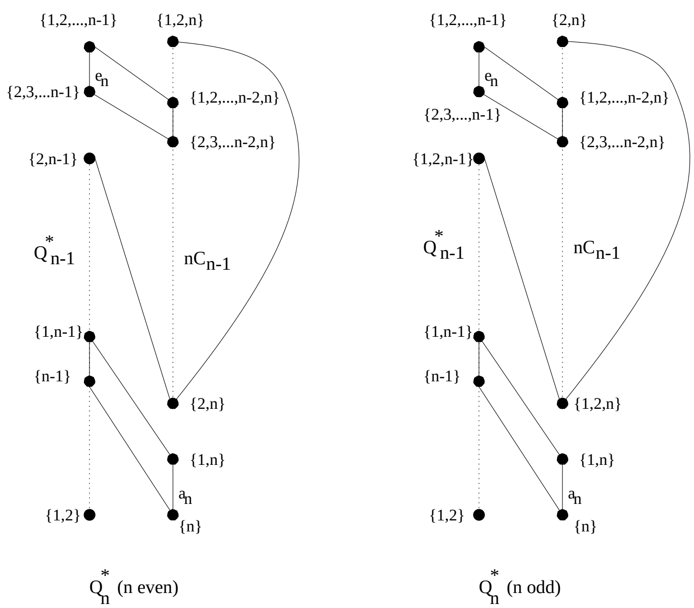

Figure 8: Construction of Q_n^* in Lemma 4.

Lemma 4 *For $n \geq 5$, G_n has a Hamiltonian path Q_n^* from $\{1,2\}$ to $\{1,2,n\}$ if n is odd and from $\{1,2\}$ to $\{2,n\}$ if n is even.*

Proof. For $n = 5$, a path Q_5^* satisfying the requirements is shown in Figure 7 (compare with Figure 2). For $n > 5$, Q_n^* is constructed from Q_{n-1}^* and nC_{n-1} exactly as Q_n is constructed from Q_{n-1} and nC_{n-1} in the proof of Lemma 3. (See Figure 8). 2

Lemma 5 *For $n \geq 6$, G_n has a Hamiltonian path P_n from $\{2\}$ to $\{1,3,4,\ldots,n\}$ if $\binom{n+1}{2}$ is odd and from $\{2\}$ to $\{3,4,\ldots,n\}$ if $\binom{n+1}{2}$ is even. In addition, when $n \geq 6$ and $\binom{n+1}{2}$ is odd, G_n has a Hamiltonian path R_n from $\{1,2\}$ to $\{3,4,\ldots,n\}$.*

Proof. For the basis case $n = 6$, paths P_6 and R_6 satisfying the requirements are given in

P_6 =

{2},	{1, 2},	{1, 3},	{3},	{4},	{1, 4},	{2, 4},
{2, 3},	{1, 2, 3},	{1, 2, 4},	{1, 2, 5},	{2, 5},	{1, 5},	{5},
{6},	{1, 6},	{2, 6},	{1, 2, 6},	{1, 3, 6},	{3, 6},	{4, 6},
{5, 6},	{1, 5, 6},	{2, 5, 6},	{1, 2, 5, 6},	{1, 2, 4, 6},	{1, 2, 3, 6},	{2, 3, 6},
{2, 4, 6},	{1, 4, 6},	{1, 4, 5},	{4, 5},	{3, 5},	{3, 4},	{1, 3, 4},
{1, 3, 5},	{2, 3, 5},	{2, 3, 4},	{1, 2, 3, 4},	{1, 2, 3, 5},	{1, 2, 4, 5},	{2, 4, 5},
{3, 4, 5},	{1, 3, 4, 5},	{2, 3, 4, 5},	{1, 2, 3, 4, 5},	{1, 2, 3, 4, 6},	{2, 3, 4, 6},	{1, 3, 4, 6},
{3, 4, 6},	{3, 5, 6},	{4, 5, 6},	{1, 4, 5, 6},	{1, 3, 5, 6},	{2, 3, 5, 6},	{1, 2, 3, 5, 6},
{1, 2, 4, 5, 6},	{2, 4, 5, 6},	{3, 4, 5, 6},	{1, 3, 4, 5, 6}			

R_6 =

{1, 2},	{2},	{3},	{4},	{5},	{6},	{1, 6},
{2, 6},	{1, 2, 6},	{1, 2, 5},	{2, 5},	{1, 5},	{1, 4},	{1, 3},
{2, 3},	{1, 2, 3},	{1, 2, 4},	{2, 4},	{3, 4},	{1, 3, 4},	{2, 3, 4},
{1, 2, 3, 4},	{1, 2, 3, 5},	{1, 2, 3, 6},	{1, 2, 4, 6},	{1, 2, 5, 6},	{2, 5, 6},	{1, 5, 6},
{5, 6},	{4, 6},	{3, 6},	{1, 3, 6},	{1, 4, 6},	{2, 4, 6},	{2, 3, 6},
{2, 3, 5},	{1, 3, 5},	{3, 5},	{4, 5},	{1, 4, 5},	{2, 4, 5},	{1, 2, 4, 5},
{1, 3, 4, 5},	{3, 4, 5},	{3, 4, 6},	{1, 3, 4, 6},	{1, 3, 5, 6},	{3, 5, 6},	{4, 5, 6},
{1, 4, 5, 6},	{2, 4, 5, 6},	{2, 3, 5, 6},	{2, 3, 4, 6},	{2, 3, 4, 5},	{1, 2, 3, 4, 5},	{1, 2, 3, 4, 6},
{1, 2, 3, 5, 6},	{1, 2, 4, 5, 6},	{1, 3, 4, 5, 6},	{3, 4, 5, 6}			

Figure 9: The Hamiltonian paths P_6 and R_6 (read across) in Lemma 5 .

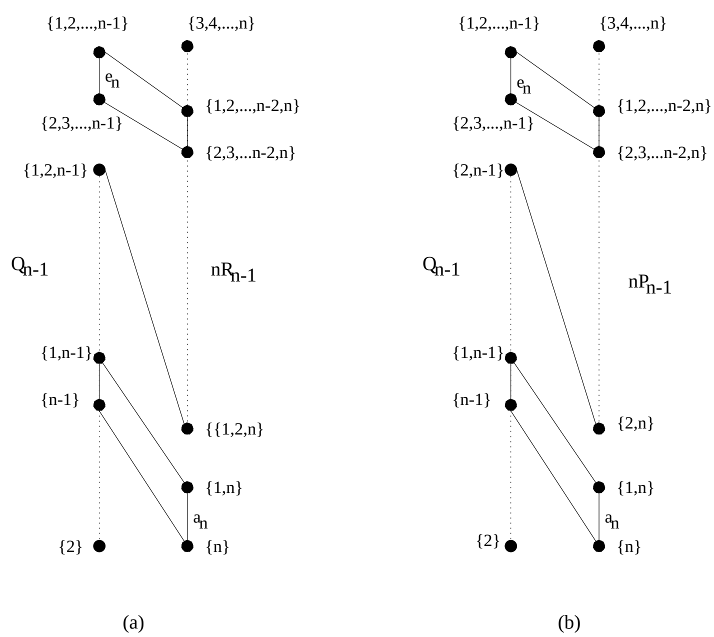

Figure 10: Construction of P_n in Lemma 5 when $\binom{n+1}{2}$ is even and n is (a) odd, (b) even.

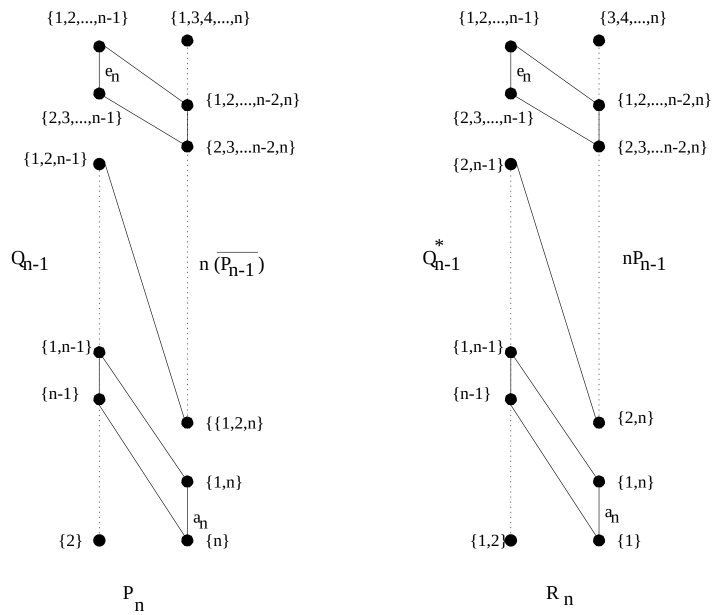

Figure 11: Construction of P_n and R_n in Lemma 5 when $\binom{n+1}{2}$ is odd and n is odd.

Figure 9. Assume $n > 6$. We consider four cases, according to whether $\binom{n+1}{2}$ and n are, independently, odd or even.

Case 1. If $\binom{n+1}{2}$ is even and n is odd, then $\binom{(n-1)+1}{2}$ is odd. By Lemma 3, G_{n-1} has a Hamiltonian path Q_{n-1} from $\{2\}$ to $\{1, 2, n-1\}$. By the induction hypothesis, nG_{n-1} has a Hamiltonian path nR_{n-1} from $\{1, 2, n\}$ to $\{3, 4, \ldots, n-1, n\}$. Construct P_n by pulling edge e_n into nR_{n-1} and edge a_n into Q_{n-1}. Now link Q_{n-1} to nR_{n-1} by adding edge $\{1, 2, n-1\}\{1, 2, n\}$. See Figure 10(a).

Case 2. If $\binom{n+1}{2}$ is even and n is even, then by Lemma 3, G_{n-1} has a Hamiltonian path Q_{n-1} from $\{2\}$ to $\{2, n-1\}$. By the induction hypothesis, nG_{n-1} has a Hamiltonian path nP_{n-1} from $\{2, n\}$ to $\{3, 4, \ldots, n-1, n\}$. Construct Hamiltonian path P_n in G_n by pulling edge e_n into nP_{n-1} and edge a_n into Q_{n-1} and then adding edge $\{2, n-1\}\{2, n\}$. See Figure 10(b).

Case 3. If $\binom{n+1}{2}$ is odd and n is odd, then we must construct both P_n and R_n. We first construct P_n. By Lemma 3, G_{n-1} has a Hamiltonian path Q_{n-1} from $\{2\}$ to $\{1, 2, n-1\}$. By the induction hypothesis, nG_{n-1} has a Hamiltonian path nP_{n-1} from $\{2, n\}$ to $\{3, 4, \ldots, n\}$. We take the complement in $[n-1]$ of every set on the path P_{n-1} to get a path $\overline{P_{n-1}}$ from $\{1, 3, 4, \ldots, n-1\}$ to $\{1, 2\}$. Now $n\overline{P_{n-1}}$ is a Hamiltonian path in nG_{n-1} from $\{1, 3, 4, \ldots, n-1, n\}$ to $\{1, 2, n\}$. Note that $n\overline{P_{n-1}}$ must have edge $\{1, 2, \ldots, n-2, n\}\{2, 3, \ldots, n-2, n\}$ by Lemma 1. To construct P_n, pull edge a_n into Q_{n-1} and edge e_n into $n\overline{P_{n-1}}$, and then join the two paths with the edge from $\{1, 2, n-1\}$ to $\{1, 2, n\}$ (Figure 11).

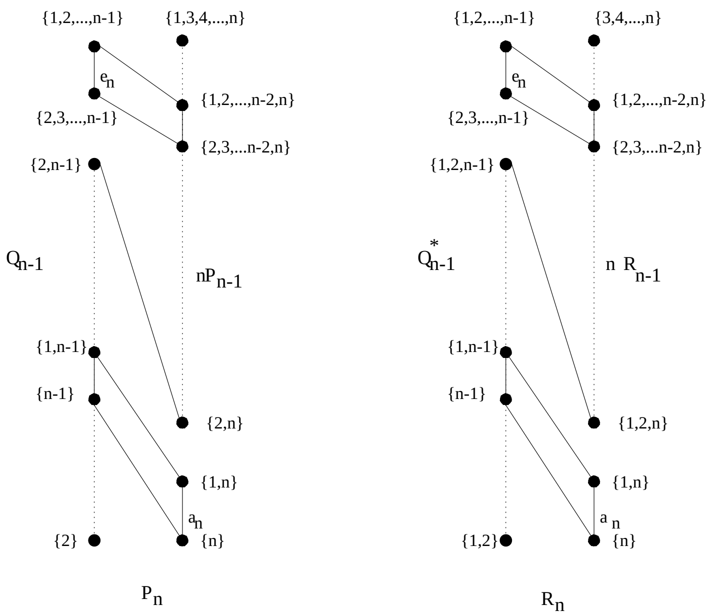

Figure 12: Construction of P_n and R_n in Lemma 5 when $\binom{n+1}{2}$ is odd and n is even.

Next we construct R_n. By Lemma 4, G_{n-1} has a Hamiltonian path Q^*_{n-1} from $\{1,2\}$ to $\{2,n-1\}$. By the induction hypothesis, nG_{n-1} has a Hamiltonian path nP_{n-1} from $\{2,n\}$ to $\{3,4,\ldots,n-1,n\}$. Pull edge e_n into nP_{n-1} and edge a_n into Q^*_{n-1}, and then join the paths by adding the edge $\{2,n-1\}\{2,n\}$ (Figure 11).

Case 4. If $\binom{n+1}{2}$ is odd and n is even, then we again construct P_n and then R_n. By Lemma 3, G_{n-1} has a Hamiltonian path Q_{n-1} from $\{2\}$ to $\{2,n-1\}$. By the induction hypothesis, nG_{n-1} has a Hamiltonian path nP_{n-1} from $\{2,n\}$ to $\{1,3,4,\ldots,n-1,n\}$. Pull edge e_n into nP_{n-1} and edge a_n into Q_{n-1}, and then join the paths by adding the edge $\{2,n-1\}\{2,n\}$ to form P_n (Figure 12).

By Lemma 4, G_{n-1} has a Hamiltonian path Q^*_{n-1} from $\{1,2\}$ to $\{1,2,n-1\}$. By the induction hypothesis, nG_{n-1} has a Hamiltonian path nR_{n-1} from $\{1,2,n\}$ to $\{3,4,\ldots,n-1,n\}$. Pull edge e_n into nR_{n-1} and edge a_n into Q^*_{n-1}, and then join the paths by adding the edge $\{1,2,n-1\}\{1,2,n\}$ to form R_n (Figure 12). *2*

Theorem 1 now follows; when $\binom{n+1}{2}$ is odd, inserting $\emptyset, \{1\}$ at the beginning and appending $\{2,3,\ldots,n\}, \{1,2,\ldots,n\}$ to the end of Q_n completes a Hamiltonian path in the full graph A_n.

The Hamiltonian path software described in [8] was most useful in allowing us initially to test the claim of Theorem 1 and then to find candidate Hamiltonian paths in G_n for the inductive construction.

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